On the Wave Characteristics of Falling Films

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The numerous studies of wavy flow characteristics of falling films testify to the complexity of the problem. One of the established schools of thought initiated by Kapitza (*8*) in 1948, and partially supported by experimental results, is based on finding a steady periodic solution to the equation of motion, by averaging the Navier-Stokes equation over the film width. However, as shown by Yih (*16*), film stability is controlled by surface waves at small Reynolds numbers. Following this idea, Anshus and Goren (*1*) modified the Orr-Sommerfeld equation, thus obtaining an analytic solution which is in excellent agreement with the exact numerical solution of Sterling and Towell (*14*), and Sterling and Barr-David (*13*). Their results are also in agreement with Jones and Whitaker's (*7*) data on wave celerity. It would thus seem that surface waves, or else the interfacial velocity, is the controlling factor affecting the film flow characteristics at low Reynolds numbers.

The present paper is an extension of Massot, Irrani, and Lightfoot's (*11*) recent solution for a steady periodic wave motion. Here, however, the normal stress induced by viscosity is included in the boundary condition and the Navier-Stokes equation is solved for by utilizing the controlling interfacial velocities rather than by averaging the equation.

FORMULATION AND SOLUTION

With reference to Figure 1, the governing equations for the two dimensional flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + g \sin\beta \tag{2}$$

subject to the boundary conditions

$$\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\bigg|_{y=h} = 0 \quad \text{(tangential stresses)} \tag{3}$$

$$-P|_{y=h} = -2\mu\frac{\partial v}{\partial y} + P_\sigma \approx -2\mu\frac{\partial v}{\partial y} + \sigma\frac{\partial^2 h}{\partial x^2} \quad \text{(normal stresses)} \tag{4}$$

where σ is the interfacial surface tension, and

$$u|_{y=o} = v|_{y=o} = 0 \tag{5}$$

It is noted that by order of magnitude considerations the y-directed equation of motion may be neglected.

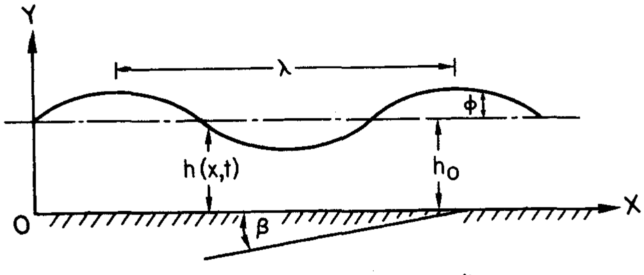

Fig. 1. Schematic presentation of coordinates.

Following common practice (*8*), the velocity distribution in the y-direction is assumed to be quasiparabolic:

$$u(x, y, t) = 3\bar{u}(x, t)\left(\frac{y}{h} - \frac{y^2}{2h^2}\right) \tag{6}$$

where $\bar{u}$ is the local mean velocity. Equations (1) and (6) yield:

$$v(x, y, t) = -3\left[\frac{\partial \bar{u}}{\partial x}\left(\frac{y^2}{2h^2} - \frac{y^3}{6h^3}\right) - \bar{u}\frac{\partial h}{\partial x}\left(\frac{y^2}{2h^2} - \frac{y^3}{3h^3}\right)\right] \tag{7}$$

Clearly, the assumption of a parabolic velocity profile in the film requires a suitable reduction in the number of equations describing the problem. The most convenient way is to eliminate the least important equation. Here, Equation (3) is reduced to the following familiar form:

$$\frac{\partial u}{\partial y}\bigg|_{y=h} = 0 \tag{8}$$

As will be shown in the discussion, this form introduces only a minor error.

Utilizing the interfacial velocities ($u \simeq u|_{y=h}$; $v \simeq v|_{y=h}$) and their related derivatives, Equations (4) and (1) yield:

$$-\frac{\partial P}{\partial x}\bigg|_{y=h} = -2\mu\left[-\frac{3}{2}\frac{\partial^2 \bar{u}}{\partial x^2}\right] + \sigma\frac{\partial^3 h}{\partial x^3} \tag{9}$$

and Equation (2) now takes the following form:

$$\frac{3}{2}\frac{\partial \bar{u}}{\partial t} + \frac{9}{4}\bar{u}\frac{\partial \bar{u}}{\partial x} = \frac{9}{2}\nu\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\sigma}{\rho}\frac{\partial^3 h}{\partial x^3} - \frac{3\bar{u}}{h^2} + g \sin\beta \tag{2a}$$

while the continuity equation leads to

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\int_o^h u\,dy = -\frac{\partial}{\partial x}(\bar{u}\,h) \tag{10}$$

For a periodic wave having a forward characteristic (that is, $z = x - ct$) one may write the relation

$$\frac{\partial \bar{u}}{\partial t} = -c\frac{\partial \bar{u}}{\partial x} \tag{11}$$

where c is the wave celerity. Expressing the film thickness in terms of the average film thickness h_o and the local amplitude, ϕ:

$$h = h_o(1 + \phi) \tag{12}$$

and utilizing Equations (10) and (11) yields (*10*):

$$u = \frac{c\phi + U_o}{1 + \phi} \tag{13}$$

where U_o is the average velocity at film width h_o. For $\phi < 1$, $\bar{u}$ and its derivatives may be approximated by the following series (*10*):

$$\bar{u} \simeq U_o + (c - U_o)\phi + (c - U_o)\phi^2 \tag{13a}$$

$$\frac{\partial \bar{u}}{\partial x} \simeq (c - U_o)(1 - 2\phi)\frac{\partial \phi}{\partial x} \tag{14}$$

$$\frac{\partial^2 \bar{u}}{\partial x^2} \simeq (c - U_o)\left[(1 - 2\phi)\frac{\partial^2 \phi}{\partial x^2} - 2\left(\frac{\partial \phi}{\partial x}\right)^2\right] \tag{15}$$

Substituting Equations (11), (12), (13a), (14), and (15) in (2a), and neglecting terms of second order, one gets:

$$\frac{\sigma}{\rho} h_o \frac{d^3\phi}{dz^3} + \frac{9}{2} \nu (c - U_o) \frac{d^2\phi}{dz^2} + \left(\frac{3}{2} c^2 - \frac{15}{4} c U_o + \frac{9}{4} U_o^2 \right) \frac{d\phi}{dz} + 3\left(g \sin \beta - \frac{\nu c}{h_o^2} \right) \phi = - g \sin \beta + \frac{3\nu U_o}{h_o^2} \quad (16)$$

In order to obtain a periodic solution of Equation (16), the following relation must exist:

$$g \sin \beta = 3\nu \frac{U_o}{h_o^2} \quad (17)$$

Note that by the first approximation the average thickness is similar to the well-known relation obtained by Nusselt (*12*).

In terms of the dimensionless wave celerity, α, Equation (16) now becomes

$$\frac{\sigma}{\rho} h_o \frac{d^3\phi}{dz^3} + \frac{9}{2} \nu U_o (\alpha - 1) \frac{d^2\phi}{dz^2} + U_o^2 \left(\frac{3}{2} \alpha^2 - \frac{15}{4} \alpha + \frac{9}{4} \right) \frac{d\phi}{dz} + \frac{U_o 3\nu}{h_o^2} (3 - \alpha)\phi = 0 \quad (18)$$

Equation (18) has a periodic steady solution only if (*11*):

$$\frac{(3 - \alpha) \dfrac{U_o 3\nu}{h_o^2} \cdot \dfrac{\sigma h_o}{\rho}}{\dfrac{9}{4} U_o \nu (\alpha - 1) U_o^2 \left(\dfrac{3}{2} \alpha^2 - \dfrac{15}{4} \alpha + \dfrac{9}{4} \right)} = 1 \quad (19)$$

In terms of the Weber number, Equation (16) becomes:

$$N_{We} \equiv \frac{U_o^2 h_o \rho}{\sigma} = \frac{4}{9} \; \frac{(3 - \alpha)}{(\alpha - 1)\left(\alpha^2 - \dfrac{5}{2} \alpha + \dfrac{3}{2} \right)} \quad (20)$$

One of the possible periodic solutions to Equation (18) is:

$$\phi = \delta \sin (mz) \quad (21)$$

where $m = 2\pi/\lambda$ is the wave number. Equation (18) and (21) yield:

$$\lambda = \frac{6\pi h_o \sqrt{\alpha - 1}}{\sqrt{\dfrac{3}{2} (3 - \alpha)}} \quad (22)$$

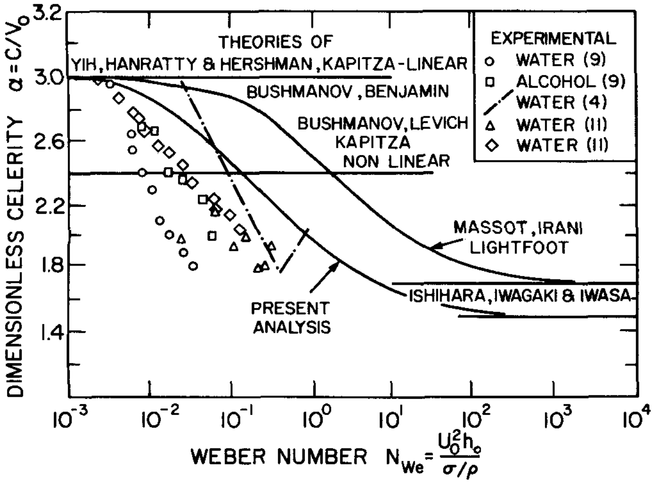

Fig. 2. Dimensionless wave celerity as a function of the Weber number.

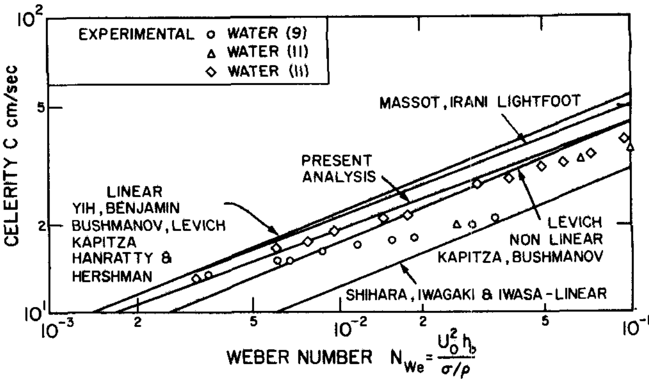

Fig. 3. Wave celerity as a function of the Weber number.

Introducing Equations (21) and (13a) into Equations (6) and (7), yield the velocities in the wavy film:

$$u(x, y, t) = 3U_o [1 + (\alpha - 1) \{\delta \sin m(x - ct) - \delta^2 \sin^2 m(x - ct) + \ldots\}] \left(\frac{y}{h} - \frac{y^2}{2h^2} \right) \quad (23)$$

$$v(x, y, t) = -3U_o (\alpha - 1) \delta m \cos m(x - ct) \left(\frac{y^2}{2h} - \frac{y^3}{6h^2} \right) + 3U_o \delta m \cos m(x - ct) \left(\frac{y^2}{2h^2} - \frac{y^3}{3h^3} \right) + \ldots \quad (24)$$

RESULTS AND CONCLUSIONS

Before proceeding with the results we will estimate the error introduced by reducing and modifying Equation (3). This is easily accomplished by analyzing Equation (24), from which one notes that $\partial v/\partial x = 0(\delta/\lambda^2)$. Since practically all the theories, including this analysis, are limited to cases where $\lambda >> \delta$, the error introduced by neglecting this derivative is negligible for such cases.

The success of the above analysis is demonstrated in Figures 2 and 3, where the values of the dimensionless wave celerity and the celerity vs. the Weber number, obtained in this analysis, were plotted on the corresponding figures reported elsewhere (*11*). The improvement over previous steady periodic solutions is better by an order of magnitude.

It is noted that the linear stability analyses of Anshus and Goren (*1*), and Whitaker (*15*) are in good agreement with the limited experimental data, shown in Figure 2, taken at relatively low Reynolds numbers, whereas the present approximate solution falls within Fulford's (*4*) numerous experimental results. (The transformation from Reynolds number to the Weber number is made by noting that for a given system $N_{We} = N_{Re}/N_\zeta$, taking $N_\zeta \simeq 5{,}000$ for water.) Moreover, the present solution though admittedly approximate is analytic throughout, and general, representing a unique relationship between the wave celerity and the Weber number, whereas the numerical stability analysis yields a particular curve for each Reynolds number, since the Weber number is introduced through the boundary condition in the solution of the Orr-Sommerfield equation.

The asymptotic values for low and high Weber numbers can be determined by Equation (20). For $N_{We} \to 0$, the value of $\alpha = 3$ is obtained, consistent with all previous information. For $N_{We} \to \infty$ we get $\alpha = 1.5$, which is in better agreement with experimental results (*4, 7*) than the value of 1.689 reported in (*11*). [$\alpha = 1$ is excluded as a solution since, from Equation (21), it would yield a wave length $\lambda = 0$ which has no physical meaning. Also, values of $1 < \alpha < 1.5$ yield negative Weber numbers, which are obviously meaningless.]

The results of this, as well as Anshus and Goren's (*1*), analysis thus substantiate Yih's (*16*) analysis regarding the controlling effect of surface waves and interfacial velocities on film characteristics.

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NOTATION

c = wave celerity
g = acceleration due to gravity
h = varying film thickness
h_o = average film thickness in wavy flow
m = wave number $(2\pi/\lambda)$
P = pressure
P = capillary pressure
t = time
u = x component of velocity
$\bar{u}$ = average film velocity at film thickness h
U_o = average film velocity at film thickness h_o
v = y-component of velocity
N_{We} = Weber number $(U_o^2 h_o \rho/\sigma)$
N_ζ = dimensionless surface tension parameter, $(\sigma 3^{1/3}/\rho g^{1/3}\nu^{4/3})$
x = cartesian coordinate
y = cartesian coordinate
z = wave forward characteristic $(x - ct)$

Greek Letters

α = dimensionless wave celerity
β = inclination angle of flow to horizontal
δ = dimensionless wave amplitude
λ = wave length
μ = dynamic viscosity
ν = kinematic viscosity
ρ = density
σ = surface tension
ϕ = local deformation of the free surface (see Figure 1)

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Relationships Between Process Equations for Processes in Connection With Newtonian and Non-Newtonian Substances

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Processes in which non-Newtonian substances take part are steadily gaining importance in process-engineering research (*1* to *7*); both the theoretical investigations and the discussion of experimental results being almost exclusively based on the friction law by Ostwald-de Waele. The numerous successes achieved in this connection should not, however, deceive us and we should recognize the limitations of this conception. Relatively simple problems, such as drag in pipelines or stirrer efficiency, are described more or less adequately with this scalar power law. Its insufficiency becomes more apparent as the processes to be investigated become more subtle. Thus, for instance, mixing operations in elastoviscous liquids can only be interpreted by complicated rheological equations of state (*8* to *10*). The endeavor of putting such operations on a better founded theoretical basis is connected with two difficulties. Firstly, the allocation of adequate rheological equations of state to real non-Newtonian substances is possible only in a few cases and secondly, theoretical discussions of such problems usually encounter great mathematical difficulties.

For this reason, the experimental research in process engineering based on similarity methods gains importance. In this connection it would be desirable to find certain universally valid rules for experimental research which are largely independent of the individual properties of the non-Newtonian substances. This includes the following *problem: A process in which a Newtonian liquid takes* part can be described by a process equation $f_{\text{Newt}}(\pi_1, \pi_2, \ldots, \pi_n) = 0$ in which π_i $(i = 1, 2, \ldots, n)$ stands for dimensionless variables due to the problem [pi-variables (*11*)]. The subscript (Newt) of the function symbol indicates that the process takes place in connection with a Newtonian liquid. A corresponding process, with otherwise unchanged conditions, may be carried out in connection with any non-Newtonian substance. It may be described by dimensionless variables π_j^* $(j = 1, 2, \ldots, m)$ and a corresponding process equation $f_{n\text{-Newt}}(\pi_1^*, \pi_2^*, \ldots, \pi_m^*) = 0$. The subscript of the function symbol indicates the non-Newtonian behavior of the substance concerned. Both sets of pi-variables form a complete set of independent quantities $S_{\text{Newt}}(\pi_i)$ and $S_{n\text{-Newt}}(\pi_j^*)$. Each set can be